

# A POSTERIORI ERROR ESTIMATION FOR THE $P$ -CURL PROBLEM\*

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**Abstract.** We derive a posteriori error estimates for a semi-discrete finite element approximation of a nonlinear eddy current problem arising from applied superconductivity, known as the  $p$ -curl problem. In particular, we show the reliability for non-conforming Nédélec elements based on a residual type argument and a Helmholtz-Weyl decomposition of  $W_0^p(\text{curl}; \Omega)$ . As a consequence, we are also able to derive an a posteriori error estimate for a quantity of interest called the AC loss. The nonlinearity for this form of Maxwell's equation is an analogue of the one found in the  $p$ -Laplacian. It is handled without linearizing around the approximate solution. The non-conformity is dealt by adapting error decomposition techniques of Carstensen, Hu and Orlando. The semi-discrete formulation studied in this paper is often encountered in commercial codes and is shown to be well-posed. The paper concludes with numerical results confirming the reliability of the a posteriori error estimate.

**Key words.** finite element, a posteriori, error estimation, Maxwell's equations, nonconforming, nonlinear, Nédélec element,  $p$ -curl problem, eddy current, divergence free

**AMS subject classifications.** 35K65, 65M60, 65M15, 78M10

**1. Introduction.** Optimal designs of next generation of high-temperature superconductor (HTS) devices will require fast and accurate approximations of the time-dependent magnetic field inside complex domains [19]. Potential devices include, among others, passive current-fault limiters, MagLev trains and power links in the CERN accelerator. In a superconductor, any reversal of variation rate in the magnetic field generates a strong front in the current density profile, as well as a discontinuity in the magnetic field profile, which is not traditionally encountered in computational electromagnetism. It is therefore clear that a posteriori error estimators can play an important role in the simulation of such devices; first to achieve design tolerances and secondly to implement adaptive mesh refinement.

At power frequencies of the applications concerned, and when the operating conditions are such that we do not exceed significantly the critical current of superconducting wires, the eddy current problem with the so-called power-law model for the resistivity adequately describes the evolution of the magnetic field  $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$  for  $(t, \mathbf{x}) \in I \times \Omega \subset \mathbb{R}^+ \times \mathbb{R}^3$  by

$$(1) \quad \partial_t \mathbf{u} + \nabla \times [\rho(\nabla \times \mathbf{u}) \nabla \times \mathbf{u}] = \mathbf{f}, \quad \text{in } I \times \Omega,$$

$$(2) \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } I \times \Omega,$$

where  $\mathbf{f}$  is known and the resistivity  $\rho$  is modeled by

$$(3) \quad \rho = \alpha |\nabla \times \mathbf{u}|^{p-2},$$

for some positive material properties  $\alpha$  and  $p$  typically between 20 and 100. The model also includes initial conditions  $\mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot)$  and boundary conditions. Although the boundary conditions are often imposed indirectly by means of a global current

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constraint, this work will focus on straightforward, but more restrictive, tangential boundary conditions

$$\mathbf{n} \times \mathbf{u} = \mathbf{g}, \quad \text{over } I \times \partial\Omega,$$

where  $\mathbf{n}$  is the exterior normal along the boundary. For consistency, the initial conditions  $\mathbf{u}_0$  and the source term  $\mathbf{f}$  must be divergence free. The precise assumptions leading to this model can be found in [24] and a description of how the above model relates to other microscopic models can be found in [10].

There is an obvious analogy between the operator  $\nabla \times (|\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u})$  of the model (1) and the  $p$ -Laplacian, namely  $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ . Researchers, Yin [38, 39], as well as Miranda, Rodrigues and Santos [27] have exploited this analogy in order to construct a well-posedness theory for the continuous problem. The key parts of that theory is the observation that the  $p$ -curl is monotone and the domain must have a smooth boundary. Formal convergence as  $p \rightarrow \infty$  of the power-law model to the Bean model has also been established in 2D [5] and in 3D [40]. Smoothness of the boundary is an essential constraint coming from the harmonic analysis in  $W^{1,p}$  spaces [22, 28, 31].

As far as we know, the theory of convergence of finite element approximation using Nédélec elements, within the same  $W^{1,p}$  framework of Yin, has yet to be established. On the other hand, using an electric field formulation of the  $p$ -curl problem, Slodička and Janíková showed convergence results within  $L^2$  spaces for backward Euler semi-discretizations and fully discretizations using linear Nédélec elements in [34, 20, 21]. However, their work have only focused on a priori error estimates.

The main result of this paper, an a posteriori error estimate, appears to be the first residual-based error estimate for the problem (1). In the work of Sirois et al. [33], adaptive time-stepping of an explicit scheme was handled by SUNDIALS [23] which contains sophisticated but generic error control strategies. The error estimates presented in this paper are residual based and resemble the a posteriori error estimators one finds for linear or linearized problems [35]. In fact, our results differ from those of Verfürth in our treatment of the non-conformity of the approximation and in our circumvention of linearization. Error estimation for FE approximate solutions of the  $p$ -Laplacian is quite well-developed and in fact, we mention the recent important work on reliable and efficient error estimation using quasi-norms [25, 8, 9, 12, 6]. The error estimate presented here also controls the error in an important quantity of interest, the AC loss over one cycle. Also, we have included a proof of the well-posedness for the straightforward semi-discretization often considered within the engineering community. Numerical results are presented to assess the quality of the error estimators. These experiments confirm the reliability of the error estimators on a class of moving front solutions in 2d.

The novelty of this paper is the treatment of the lack of conformity of the Nédélec element approximations. Inspired largely by the work of Carstensen, Ju and Orlando on the issue [7], we have found that coercive estimates are sufficient to obtain reliable error estimates. This is in stark contrast to most nonlinear problems which require a linearization of the operator in a neighborhood of the numerical solution. Given that the semi-discretization considered here is also found in commercial codes, and that the a posteriori error estimators of this paper are straightforward to implement, it appears that this work could be of interest to the engineering community.

The paper is organized as follows. The second section presents a brief review of the functional analysis required for the a posteriori error estimation. In Section 3, for the sake of completeness we include a demonstration of the well-posedness of our

semi-discretization of the  $p$ -curl problem. The fourth section contains the proof of the main theorem. It is later extended in Section 5 to the control of the AC loss. The last section describes numerical results obtained when comparing the error estimator to the exact error for a class of moving front solutions using the method of manufactured solutions and as well as convergence results for a backward Euler discretization. In Appendix A, we have extended the a posteriori error estimator to the case of non-homogeneous tangential boundary conditions, exploiting again properties unique to the  $p$ -Laplacian and the  $p$ -curl problem.

**2. Preliminaries.** This section reviews the main functional spaces over which the  $p$ -curl problem is examined and it states the strong and weak forms of the problem. It concludes with a detailed presentation of the two main technical tools, namely the Helmholtz-Weyl decomposition over  $L^p$  spaces and the quasi-interpolation operator of Schöberl [30].

Let  $\Omega$  be a bounded  $C^{1,1}$  domain, that is to say that at each point on the boundary, there exists a neighborhood of the form  $V \times (a, b) \subset \mathbb{R}^{d-1} \times \mathbb{R}$  and a  $C^1$  function  $\psi : V \rightarrow \mathbb{R}$  with Lipschitz continuous derivatives such that  $\Omega \cap V \times (a, b) = \{\mathbf{x} | \psi(\mathbf{x}) < 0\}$ . Let  $k$  be a nonnegative integer and for  $s \geq 0$  denote its integer part as  $[s]$ . Throughout, we denote  $q$  as the Hölder conjugate exponent of  $p$  satisfying  $1 = 1/p + 1/q$ . Recall the following well-known Sobolev spaces [1].

$$\begin{aligned} W^{k,p}(\Omega) &= \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega)^d, |\alpha| \leq k\} \\ W^{s,p}(\Omega) &= \left\{ v \in W^{[s],p}(\Omega) : \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|=[s]}} \left\| \frac{D^\alpha v(x) - D^\alpha v(y)}{|x-y|^{d/p+s-[s]}} \right\|_{L^p(\Omega \times \Omega)}^p < \infty, \right\} \\ W_0^{s,p}(\Omega) &= \{v \in W^{s,p}(\Omega) : \gamma_0(v) = 0\} \\ W^{-s,p}(\Omega) &= (W_0^{s,q}(\Omega))' \end{aligned}$$

For our problem, minimal regularity suggests that we consider the following spaces; see [28, 3] for more details on their properties and equivalent norms.

$$\begin{aligned} W^p(\text{curl}; \Omega) &= \{\mathbf{v} \in L^p(\Omega)^d : \nabla \times \mathbf{v} \in L^p(\Omega)^d\} \\ W_0^p(\text{curl}; \Omega) &= \{\mathbf{v} \in W^p(\text{curl}; \Omega) : \gamma_t(\mathbf{v}) = 0\} \\ W^p(\text{div}; \Omega) &= \{\mathbf{v} \in L^p(\Omega)^d : \nabla \cdot \mathbf{v} \in L^p(\Omega)\} \\ W^p(\text{div}^0; \Omega) &= \{\mathbf{v} \in W^p(\text{div}; \Omega) : \nabla \cdot \mathbf{v} = 0\} \\ V^p(\Omega) &= W_0^p(\text{curl}; \Omega) \cap W^p(\text{div}^0; \Omega) \end{aligned}$$

Above,  $\gamma_0 : W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)$  is the continuous boundary trace operator and  $\gamma_t : W^p(\text{curl}; \Omega) \rightarrow (W^{1-1/p,p}(\partial\Omega)^d)'$ ,  $\gamma_n : W^p(\text{div}; \Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)'$  are the continuous tangential and normal trace operators satisfying:

(4)

$$(\gamma_t(\mathbf{v}), \gamma_0(\mathbf{w}))_{\partial\Omega} = \int_{\Omega} \mathbf{v} \cdot \nabla \times \mathbf{w} \, dV - \int_{\Omega} \mathbf{w} \cdot \nabla \times \mathbf{v} \, dV, \quad \forall \mathbf{v}, \mathbf{w} \in W^p(\text{curl}; \Omega),$$

(5)

$$(\gamma_n(\mathbf{v}), \gamma_0(w))_{\partial\Omega} = \int_{\Omega} \mathbf{v} \cdot \nabla w \, dV + \int_{\Omega} \nabla \cdot \mathbf{v} w \, dV, \quad \forall \mathbf{v} \in W^p(\text{div}; \Omega), w \in W^{1,p}(\Omega).$$

For sufficiently smooth functions  $\mathbf{v}$  and  $w$ , these trace operators are simply  $\gamma_0(w) = w|_{\partial\Omega}$ ,  $\gamma_t(\mathbf{v}) = \mathbf{n} \times \mathbf{v}|_{\partial\Omega}$  and  $\gamma_n(\mathbf{v}) = \mathbf{n} \cdot \mathbf{v}|_{\partial\Omega}$ . Later, we will need the stability bound below [1].

LEMMA 1. *Let  $\Omega$  be a bounded domain with a Lipschitz boundary. If  $\mathbf{v} \in W^{1,p}(\Omega)$ , then the boundary trace operator  $\gamma_0 : W^{1,p}(K) \rightarrow L^p(\partial K)$  is a continuous linear operator, i.e. there exist a constant  $C > 0$  such that,*

$$(6) \quad \|\gamma_0(\mathbf{v})\|_{L^p(\partial K)} \leq C \|\mathbf{v}\|_{W^{1,p}(K)}.$$

As is customary for  $L^2$  spaces, we write  $W^{k,2}(\Omega)$  as  $H^k(\Omega)$  and similarly we write  $W^2(\text{div}; \Omega)$  and  $W^2(\text{curl}; \Omega)$  as  $H(\text{div}; \Omega)$  and  $H(\text{curl}; \Omega)$ , respectively.

If  $\mathbf{u} \in L^q(\Omega)^d$ ,  $\mathbf{v} \in L^p(\Omega)^d$ , we denote the pairing

$$(\mathbf{u}, \mathbf{v})_\Omega := \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dV,$$

and define the nonlinear operator  $\mathcal{P} : W^p(\text{curl}; \Omega) \rightarrow W^p(\text{curl}; \Omega)'$ ,

$$(7) \quad \langle \mathcal{P}(\mathbf{u}), \mathbf{v} \rangle_\Omega := (\rho(\nabla \times \mathbf{u}) \nabla \times \mathbf{u}, \nabla \times \mathbf{v})_\Omega.$$

Indeed, by Holder's inequality, these pairings are well-defined since,

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_\Omega &\leq \|\mathbf{u}\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^p(\Omega)}, \\ \langle \mathcal{P}(\mathbf{u}), \mathbf{v} \rangle_\Omega &\leq \|\nabla \times \mathbf{u}\|_{L^p(\Omega)}^{p/q} \|\nabla \times \mathbf{v}\|_{L^p(\Omega)}. \end{aligned}$$

Over the time interval  $I = [0, T]$ , the  $p$ -curl problem arising from applied superconductivity is the following nonlinear evolutionary equation:

$$(8) \quad \begin{aligned} \partial_t \mathbf{u} + \nabla \times [\rho(\nabla \times \mathbf{u}) \nabla \times \mathbf{u}] &= \mathbf{f}, & \text{in } I \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0, & \text{in } I \times \Omega, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0(\cdot), & \text{in } \Omega, \\ \mathbf{n} \times \mathbf{u} &= 0, & \text{on } I \times \partial\Omega, \end{aligned}$$

where  $p \geq 2$ ,  $\rho$  is the nonlinear resistivity modeled by an isotropic power law  $\rho(\nabla \times \mathbf{u}) = \alpha |\nabla \times \mathbf{u}|^{p-2}$  and  $\alpha = E_0/(\mu J_c^{p-1}) > 0$  is a material dependent constant. Moreover, it is assumed that  $\nabla \cdot \mathbf{u}_0 = 0$  and  $\nabla \cdot \mathbf{f} = 0$  for all  $t \in I$  in a manner to be made precise later.

The weak formulation of the  $p$ -curl problem is:

Given  $\mathbf{u}_0 \in W^p(\text{div}^0; \Omega)$  and  $\mathbf{f} \in L^2(I; W^q(\text{div}^0; \Omega))$ , find  $\mathbf{u} \in L^2(I; V^p(\Omega)) \cap H^1(I; L^q(\Omega))$  satisfying  $\mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot)$  and

$$(9) \quad (\partial_t \mathbf{u}, \mathbf{v})_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{v} \rangle_\Omega = (\mathbf{f}, \mathbf{v})_\Omega, \quad \forall \mathbf{v} \in L^2(I; V^p(\Omega)).$$

The well-posedness of the weak problem was established in the work of Yin et al. [39, 40].

Following the presentation of the continuous problem, we provide an overview of the method of lines discretization for the  $p$ -curl problem. We begin by discussing discretizations of space by tetrahedral meshes and finite dimensional approximations of the function spaces presented above. For the sake of simplicity, the description will

be given only in  $\mathbb{R}^3$  although the modifications to  $\mathbb{R}^2$  should be obvious. Let  $\mathcal{T}_h := \{K \subset \Omega : K \text{ a tetrahedron in } \mathbb{R}^3\}$  be a shape-regular triangularization of  $\Omega = \bigcup_{K \in \mathcal{T}_h} K$  with the obvious constraints that are required to ensure that the set of faces  $\mathcal{F}(\mathcal{T}_h)$ , the set of edges  $\mathcal{E}(\mathcal{T}_h)$ , and the set of nodes are well-defined. For each tetrahedron  $K \in \mathcal{T}_h$ , let  $h_K$  be the diameter of the largest inscribed sphere, while for each face  $F$  we can define  $h_F$  to be the diameter of the largest inscribed circle. Let  $h$  be the largest diameter over all  $K \in \mathcal{T}_h$ .

In  $\mathbb{R}^3$ , the order  $k$  Nédélec finite element space is defined as,

$$(10) \quad V_{h,0}^{(k)} := \{\mathbf{v} \in W_0^p(\text{curl}; \Omega) : \mathbf{v}|_K = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \times \mathbf{x}, \mathbf{a}, \mathbf{b} \in (\mathbb{P}_{k-1})^3, K \in \mathcal{T}_h\},$$

where  $(\mathbb{P}_k)^3$  is the space of vector fields with polynomial components of at most degree  $k$ . For fixed  $k$ , this is the space of  $k$ -th order elements of the first family of Nédélec [29]. Recall that the finite element space  $V_{h,0}^{(k)}$  is uniquely determined by identifying the degrees of freedom of the surface integral along faces and edges between any two neighboring elements. Since an element-wise  $W^p(\text{curl}; K)$  defined function that is continuous tangentially along faces and edges is a global  $W^p(\text{curl}; \Omega)$  function,  $V_{h,0}^{(k)} \subset W_0^p(\text{curl}; \Omega)$ . Moreover  $V_{h,0}^{(1)}$  is known to be locally divergence-free, i.e.  $\nabla \cdot \mathbf{v}|_K = 0$  for  $\mathbf{v} \in V_{h,0}^{(1)}$ , and thus it is an element-wise  $W^p(\text{div}^0; \Omega)$  defined function. Unfortunately, higher order elements will not be in  $W^p(\text{div}^0; \Omega)$ . In any case,  $V_{h,0}^{(k)}$  can be discontinuous in the normal direction to faces and edges and hence in general is not a global  $W^p(\text{div}; \Omega)$  function. In particular,  $V_{h,0}^{(k)} \not\subset V^p(\Omega)$ .

This following lemma is obtained by combining Lemma 1 with a standard scaling argument.

LEMMA 2. *Let  $K \in \mathcal{T}$  and  $F$  be any face of  $K$ . If  $v \in W^{1,p}(K)$ , then there exists constant  $C > 0$  so that,*

$$(11) \quad h_F^{1-p} \|\gamma_0(v)\|_{L^p(F)}^p \leq C(h_F^{-p} \|v\|_{L^p(K)}^p + \|\nabla v\|_{L^p(K)}^p),$$

where  $h_F$  is the diameter of the largest circle inscribed in  $F$ .

This leads us to the non-conforming semi-discrete weak formulation of the  $p$ -curl problem:

Given  $\mathbf{u}_{0,h} \in V_{h,0}^{(k)}$  and  $\mathbf{f} \in C(I; W^q(\text{div}^0; \Omega))$ , find  $\mathbf{u}_h \in C^1(I; V_{h,0}^{(k)})$  satisfying  $\mathbf{u}_h(0, \cdot) = \mathbf{u}_{0,h}(\cdot)$  and

$$(12) \quad (\partial_t \mathbf{u}_h, \mathbf{v}_h)_\Omega + \langle \mathcal{P}(\mathbf{u}_h), \mathbf{v}_h \rangle_\Omega = (\mathbf{f}, \mathbf{v}_h)_\Omega, \quad \forall \mathbf{v}_h \in V_{h,0}^{(k)}.$$

Due to the nonconformity, well-posedness of the semi-discretization does not necessarily follow from the well-posedness of the weak formulation. By a local existence argument and a priori estimate, the semi-discretization is shown to be well-posed in Section 3. Note that, while the weak formulation only requires  $\mathbf{f}$  to be  $L^2$  in  $t$ , we need  $\mathbf{f}$  to be continuous in  $t$  in order to apply Picard's local existence theorem.

We now proceed with a rather detailed review of the Helmholtz-Weyl decomposition for  $L^p$  spaces. This is needed in order to address the non-conformity in a similar manner as appeared in [7]. The most technical aspects concerning the  $p$ -curl problem turn out to be related to this decomposition, not only because of the Banach nature

of the  $L^p$  spaces concerned, but also because it imposes strict limits on the regularity of the boundary.

Define  $L_\sigma^p(\Omega) := \text{closure of } \{\mathbf{v} \in C_0^\infty(\Omega)^d : \nabla \cdot \mathbf{v} = 0\}$  with respect to  $L^p$  norm. Recall that we assume throughout that  $\Omega$  is a domain with a  $C^{1,1}$  boundary. A standard formulation of the decomposition is the following.

*There exists a positive constant  $C = C(\Omega, p, d)$  such that for any  $\mathbf{v} \in L^p(\Omega)^d$ , there exists  $\phi \in W^{1,p}(\Omega)/\mathbb{R}$  and  $\mathbf{z} \in L_\sigma^p(\Omega)$  for which  $\mathbf{v} = \mathbf{z} + \nabla\phi$  and*

$$(13) \quad \|\mathbf{z}\|_{L^p(\Omega)} + \|\nabla\phi\|_{L^p(\Omega)} \leq C \|\mathbf{v}\|_{L^p(\Omega)}.$$

When the vector field has zero boundary trace, then the Helmholtz-Weyl decomposition is as follows.

*There exists a positive constant  $C = C(\Omega, p, d)$  such that for any  $\mathbf{v} \in L^p(\Omega)^d$ , there exists  $\phi \in W_0^{1,p}(\Omega)$  and  $\mathbf{z} \in W^p(\text{div}^0; \Omega)$  for which  $\mathbf{v} = \mathbf{z} + \nabla\phi$  and*

$$(14) \quad \|\mathbf{z}\|_{L^p(\Omega)} + \|\nabla\phi\|_{L^p(\Omega)} \leq C \|\mathbf{v}\|_{L^p(\Omega)}.$$

While the decomposition when  $p = 2$  can be studied using tools no more complicated than the Lax-Milgram theorem, the case for general  $p$  is much more subtle. It has been observed (for example [17, Lemma III 1.2]) that the existence of the Helmholtz-Weyl decomposition of (13) is equivalent to the solvability of the following Neumann problem over  $\Omega$ .

*Given  $\mathbf{v} \in L^p(\Omega)^d$ , find  $\phi \in W^{1,p}(\Omega)/\mathbb{R}$  such that for all  $\psi \in W^{1,q}(\Omega)/\mathbb{R}$ ,*

$$(\nabla\phi, \nabla\psi)_\Omega = (\mathbf{v}, \nabla\psi)_\Omega.$$

Similarly, the existence of Helmholtz-Weyl decomposition of (14) is equivalent to the solvability of the Dirichlet problem below.

*Given  $\mathbf{v} \in L^p(\Omega)^d$ , find  $\phi \in W_0^{1,p}(\Omega)$  such that for all  $\psi \in W_0^{1,q}(\Omega)$ ,*

$$(\nabla\phi, \nabla\psi)_\Omega = (\mathbf{v}, \nabla\psi)_\Omega.$$

In particular, if  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain and  $\epsilon(\Omega) > 0$  depending Lipschitz constant of  $\Omega$ , it was shown in [14] that the above Neumann problem has a solution in the a sharp region near  $p \in (3/2 - \epsilon, 3 + \epsilon)$ . Similarly, [22] showed that the above Dirichlet problem has a solution in a sharp region near  $p \in (2/(1 + \epsilon), 2/(1 - \epsilon))$ . This implies the Helmholtz-Weyl decomposition does not hold in general for bounded Lipschitz domains, which is unfortunate since such domains do arise in engineering applications of superconductors. Thus, we are forced to restrict to bounded  $C^{1,1}$  domains, which is consistent with the regularity of the boundary required for the well-posedness of the  $p$ -curl problem given by [40].

The Helmholtz-Weyl decomposition for  $L^2$  was first demonstrated by [37] and for  $L^p$  by [16] for smooth bounded domains. To our best knowledge, results concerning minimal regularity requirement on the boundary are known for bounded  $C^1$  domains [31, 32] and more recently for bounded convex domains [18].

THEOREM 3. [32, Theorem II.1.1] Let  $\Omega \subset \mathbb{R}^d$  be bounded  $C^1$  domain and let  $1 < p < \infty$ . Then the Helmholtz-Weyl decomposition (14) holds.

THEOREM 4. [18, Theorem 1.3] Let  $\Omega \subset \mathbb{R}^d$  be bounded convex domain and let  $1 < p < \infty$ . Then the Helmholtz-Weyl decomposition (13) holds.

We also mention [3] have derived an  $L^p$  version of the Hodge decomposition for domains with  $C^{1,1}$  boundary. We now use Theorem 3 to derive a new Helmholtz-Weyl decomposition for  $W_0^p(\text{curl}; \Omega)$ .

LEMMA 5. Let  $\Omega \subset \mathbb{R}^d$  be a bounded simply connected  $C^1$  domain and let  $2 \leq p < \infty$ . Then the following direct sum holds,

$$W_0^p(\text{curl}; \Omega) = V^p(\Omega) \oplus \nabla W_0^{1,p}(\Omega).$$

In other words, for any  $\mathbf{v} \in W_0^p(\text{curl}; \Omega)$ , there exists unique  $\phi \in W_0^{1,p}(\Omega)$  and  $\mathbf{z} \in V^p(\Omega)$  such that  $\mathbf{v} = \mathbf{z} + \nabla \phi$  satisfying,

$$(15) \quad \|\mathbf{z}\|_{L^p(\Omega)} + \|\nabla \phi\|_{L^p(\Omega)} \leq C \|\mathbf{v}\|_{L^p(\Omega)}, \quad C = C(\Omega, p, d) > 0.$$

*Proof.* Let  $\mathbf{v} \in W_0^p(\text{curl}; \Omega) \subset L^p(\Omega)^d$ . Then by Theorem 3,  $\mathbf{v} = \nabla \phi + \mathbf{z}$  for some  $\phi \in W_0^{1,p}(\Omega)$  and  $\mathbf{z} \in W^p(\text{div}^0; \Omega)$ . Since  $\nabla W_0^{1,p}(\Omega) \subset W^p(\text{curl}; \Omega)$ ,  $\gamma_t(\nabla \phi)$  is well defined. Let  $\{\phi_k \in C_0^\infty(\Omega)\}$  converging to  $\phi$  in  $W_0^{1,p}(\Omega)$ . Since  $\gamma_0(\nabla \phi_k) = 0$  and so  $\gamma_t(\nabla \phi_k) = 0$ , then by continuity of the tangential trace operator  $\gamma_t(\nabla \phi) = 0$  and so  $\mathbf{z} = \mathbf{v} - \nabla \phi \in W_0^p(\text{curl}; \Omega)$ . I.e.  $\mathbf{z} \in V^p(\Omega)$ .

To show the sum is direct, suppose  $\mathbf{v} \in V^p(\Omega) \cap \nabla W_0^{1,p}(\Omega)$ . Then  $\mathbf{v} = \nabla \phi$  for some  $\phi \in W_0^{1,p}(\Omega)$ . Since  $\mathbf{v} \in V^p(\Omega)$ , for all  $\psi \in W_0^{1,q}(\Omega)$ ,

$$(16) \quad 0 = (\mathbf{v}, \nabla \psi)_\Omega = (\nabla \phi, \nabla \psi)_\Omega$$

As  $p \geq 2 \geq q > 1$ ,  $\phi \in W_0^{1,2}(\Omega) \subset W_0^{1,q}(\Omega)$ . Setting  $\psi = \phi$  in (16) implies  $\|\nabla \phi\|_{L^2(\Omega)} = 0$  and hence  $\phi = 0$  a.e. by Friedrichs' inequality. I.e.  $\mathbf{v} = \nabla \phi = 0$ .  $\square$

Finally, we conclude with the quasi-interpolation operator  $\Pi$  of Schöberl [30], which for Nédélec elements plays the same role as the Clément operator does for Lagrange elements.

THEOREM 6. There exists a quasi-interpolation operator  $\Pi : H_0(\text{curl}; \Omega) \rightarrow V_{h,0}^{(0)}$  with the property: for any  $\mathbf{v} \in H_0(\text{curl}; \Omega)$ , there exists  $\phi \in H_0^1(\Omega)$  and  $\mathbf{w} \in H_0^1(\Omega)^3$  such that,

$$(17) \quad \mathbf{v} - \Pi \mathbf{v} = \nabla \phi + \mathbf{w},$$

and on each  $K \in \mathcal{T}$ , there exists a neighbourhood  $\omega_K \subset \Omega$  of  $\bar{K}$  and a constant  $C > 0$  depending only on shape-regularity of the elements in  $\omega_K$  such that  $\phi, \mathbf{w}$  satisfy,

$$(18) \quad h_K^{-1} \|\phi\|_{L^2(K)} + \|\nabla \phi\|_{L^2(K)} \leq C \|\mathbf{v}\|_{L^2(\omega_K)},$$

$$(19) \quad h_K^{-1} \|\mathbf{w}\|_{L^2(K)} + \|\nabla \mathbf{w}\|_{L^2(K)} \leq C \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)}.$$

**3. Well-posedness of the semi-discretization.** This section contains a short proof of the well-posedness of the semi-discrete weak formulation of (12). The well-posedness is not required for the construction of the a posteriori error estimators in the following section, and so this section can be read independently of the others. Nevertheless, for the sake of accessibility, this topic is best discussed first.

**THEOREM 7.** *There exists a unique solution  $\mathbf{u}_h \in C^1(I; V_{h,0}^{(k)})$  satisfying the semi-discrete weak formulation of (12). Moreover, the stability estimates hold,*

$$(20) \quad \sup_{t \in [0, T]} \|\mathbf{u}_h(t)\|_{L^2(\Omega)}^2 + 2 \int_0^T \|\nabla \times \mathbf{u}_h(s)\|_{L^p(\Omega)}^p ds \leq e \left( \|\mathbf{u}_{0,h}\|_{L^2(\Omega)}^2 + T \int_0^T \|\mathbf{f}(s)\|_{L^2(\Omega)}^2 ds \right),$$

$$(21) \quad \int_0^T \|\partial_t \mathbf{u}_h(s)\|_{L^2(\Omega)}^2 ds + \sup_{t \in [0, T]} \|\nabla \times \mathbf{u}_h(t)\|_{L^p(\Omega)}^p \leq \|\nabla \times \mathbf{u}_{0,h}\|_{L^p(\Omega)}^p + \frac{p^2}{4(p-1)} \int_0^T \|\mathbf{f}(s)\|_{L^2(\Omega)}^2 ds.$$

*Proof.* The space of  $k$ -th order Nédélec elements  $V_{h,0}^{(k)}$  is a closed subspace of  $W^p(\text{curl}; \Omega)$  and we restrict the norm of  $W^p(\text{curl}; \Omega)$  to it,

$$\|\mathbf{v}_h\|_{W^p(\text{curl}; \Omega)}^p = \|\mathbf{v}_h\|_{L^p(\Omega)}^p + \|\nabla \times \mathbf{v}_h\|_{L^p(\Omega)}^p, \quad \mathbf{v}_h \in V_{h,0}^{(k)}.$$

By Riesz representation theorem for  $L^p$  functions, there exists an isometry  $R : L^q(\Omega) \rightarrow L^p(\Omega)'$ , also known as the Riesz map. Then we can view the semi-discrete weak formulation of (12) as seeking an unique solution  $\mathbf{u}_h \in C^1(I; V_{h,0}^{(k)})$  to the first order ODEs,

$$(22) \quad R\partial_t \mathbf{u}_h(t) = -\mathcal{P}(\mathbf{u}_h(t)) + \mathbf{f}(t).$$

The proof proceeds in 2 steps. First, we show local existence for (22). Second, we extend its interval of existence to  $I$  by a priori estimates.

To show local existence, we verify that the right hand side of (22) is continuous in  $t$  and locally Lipschitz continuous in  $\mathbf{u}_h$ . Indeed, since  $\mathbf{f} \in C(I; W^q(\text{div}^0; \Omega))$  and  $q < 2$ ,  $\mathbf{f} \in L^q(\Omega)$  for all  $t \in I$ . This implies for any  $\mathbf{v} \in W^p(\text{curl}; \Omega)$  and  $t, s \in I$ ,

$$\begin{aligned} |(\mathbf{f}(t) - \mathbf{f}(s), \mathbf{v})_\Omega| &\leq \|\mathbf{f}(t) - \mathbf{f}(s)\|_{L^q(\Omega)} \|\mathbf{v}\|_{L^p(\Omega)} \\ &\leq \|\mathbf{f}(t) - \mathbf{f}(s)\|_{L^q(\Omega)} \|\mathbf{v}\|_{W^p(\text{curl}; \Omega)}. \end{aligned}$$

It follows that,

$$\|\mathbf{f}(t) - \mathbf{f}(s)\|_{W^p(\text{curl}; \Omega)'} = \sup_{0 \neq \mathbf{v} \in W^p(\text{curl}; \Omega)} \frac{|(\mathbf{f}(t) - \mathbf{f}(s), \mathbf{v})_\Omega|}{\|\mathbf{v}\|_{W^p(\text{curl}; \Omega)}} \leq \|\mathbf{f}(t) - \mathbf{f}(s)\|_{L^q(\Omega)},$$

which tends to 0 as  $s \rightarrow t$ . This shows  $\mathbf{f}(t) \in W^p(\text{curl}; \Omega)'$  is continuous in  $t$ .

Now recall from [4, Lemma 2.2], that the following equality holds for some  $C_p > 0$ ,

$$||\mathbf{x}|^{p-2}\mathbf{x} - |\mathbf{y}|^{p-2}\mathbf{y}| \leq C_p |\mathbf{x} - \mathbf{y}| (|\mathbf{x}| + |\mathbf{y}|)^{p-2}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$



So for any  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^p(\text{curl}; \Omega)$ , it follows from the above inequality and Hölder's inequality that,

$$\begin{aligned} |\langle \mathcal{P}(\mathbf{u}) - \mathcal{P}(\mathbf{w}), \mathbf{v} \rangle_\Omega| &\leq \int_\Omega \left| |\nabla \times \mathbf{u}|^{p-2} \nabla \times \mathbf{u} - |\nabla \times \mathbf{v}|^{p-2} \nabla \times \mathbf{v} \right| |\nabla \times \mathbf{v}| dV \\ &\leq C_p \int_\Omega |\nabla \times (\mathbf{u} - \mathbf{w})| (|\nabla \times \mathbf{u}| + |\nabla \times \mathbf{w}|)^{p-2} |\nabla \times \mathbf{v}| dV \\ &\leq C_p \|\nabla \times \mathbf{v}\|_{L^p(\Omega)} \|\nabla \times (\mathbf{u} - \mathbf{w})\|_{L^p(\Omega)} \| |\nabla \times \mathbf{u}| + |\nabla \times \mathbf{w}| \|_{L^p(\Omega)}^{p-2} \\ &\leq C_p \|\mathbf{v}\|_{W^p(\text{curl}; \Omega)} \|\mathbf{u} - \mathbf{w}\|_{W^p(\text{curl}; \Omega)} \| |\nabla \times \mathbf{u}| + |\nabla \times \mathbf{w}| \|_{L^p(\Omega)}^{p-2}. \end{aligned}$$

Thus, we have that for any compact subset  $A \subset V_{h,0}^{(k)}$  and any  $\mathbf{u}_h, \mathbf{w}_h \in A$ ,

$$\begin{aligned} \|\mathcal{P}(\mathbf{u}_h) - \mathcal{P}(\mathbf{w}_h)\|_{W^p(\text{curl}; \Omega)'} &= \sup_{0 \neq \mathbf{v} \in W^p(\text{curl}; \Omega)} \frac{|\langle \mathcal{P}(\mathbf{u}_h) - \mathcal{P}(\mathbf{w}_h), \mathbf{v} \rangle_\Omega|}{\|\mathbf{v}\|_{W^p(\text{curl}; \Omega)}} \\ &\leq \left( C_p \max_{\mathbf{u}_h, \mathbf{w}_h \in A} \| |\nabla \times \mathbf{u}_h| + |\nabla \times \mathbf{w}_h| \|_{L^p(\Omega)}^{p-2} \right) \|\mathbf{u}_h - \mathbf{w}_h\|_{W^p(\text{curl}; \Omega)}. \end{aligned}$$

This shows that  $\mathcal{P}(\mathbf{u}_h)$  is locally Lipschitz continuous in  $\mathbf{u}_h$ . Thus, by Picard's existence theorem, there exists a unique local solution  $\mathbf{u}_h \in C^1([0, \tilde{T}); V_{h,0}^{(k)})$  to (12), with  $[0, \tilde{T}) \subset I$ .

Finally, we extend  $[0, \tilde{T})$  to  $I$  by showing the following a priori estimates. At every  $t \in [0, \tilde{T})$ , we have  $\mathbf{u}_h(t, \cdot) \in V_{h,0}^{(k)}$ . Setting now  $\mathbf{v}_h = \mathbf{u}_h$  in (12), Young's inequality and Gronwall's inequality imply

$$\begin{aligned} \frac{d}{dt} \|\mathbf{u}_h\|_{L^2(\Omega)}^2 + 2 \|\nabla \times \mathbf{u}_h\|_{L^p(\Omega)}^p &\leq \epsilon \|\mathbf{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{\epsilon} \|\mathbf{f}\|_{L^2(\Omega)}^2 \\ \Rightarrow \|\mathbf{u}_h(t)\|_{L^2(\Omega)}^2 + 2 \int_0^t \|\nabla \times \mathbf{u}_h(s)\|_{L^p(\Omega)}^p ds &\leq e^{\epsilon T} \|\mathbf{u}_{0,h}\|_{L^2(\Omega)}^2 + \frac{e^{\epsilon T}}{\epsilon} \int_0^t \|\mathbf{f}\|_{L^2(\Omega)}^2 ds \end{aligned}$$

Thus, taking supremum on the left hand side and setting  $\epsilon = \frac{1}{\tilde{T}}$  shows the stability estimate (20), which implies  $[0, \tilde{T})$  can be extended to  $I$ . Similarly, the second stability estimate (21) follows by setting  $\mathbf{v}_h = \partial_t \mathbf{u}_h$  in (12) and noting that  $\frac{1}{p} \frac{d}{dt} \|\nabla \times \mathbf{u}_h\|_{L^p(\Omega)}^p = \langle \mathcal{P}(\mathbf{u}_h), \partial_t \mathbf{u}_h \rangle_\Omega$ .  $\square$

**4. A posteriori error estimator.** This section contains the main result of this paper, Theorem 10. The proof follows the usual residual-based approach except for the treatment of the nonconformity and nonlinearity. We begin with Lemma 8 which is essentially equivalent to Galerkin orthogonality. This is then used to bound the error, as stated in Theorem 9. Afterwards, stability estimates for both the trace operator and the Schöberl's quasi-interpolation operator allow us to combine the local estimate into a global estimate of Theorem 10.

**LEMMA 8.** *Consider a  $C^1$  simply connected bounded domain  $\Omega$  and a source term  $\mathbf{f} \in L^2(I; W^q(\text{div}^0; \Omega))$ . Assume that  $\mathbf{u}$  is a weak solution to (9), then*

$$(23) \quad (\partial_t \mathbf{u}, \mathbf{v})_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{v} \rangle_\Omega = (\mathbf{f}, \mathbf{v})_\Omega, \quad \forall \mathbf{v} \in W_0^p(\text{curl}; \Omega).$$

*Proof.* Let  $\mathbf{v} \in W_0^p(\text{curl}; \Omega)$ . By Lemma 5,  $\mathbf{v} = \mathbf{z} + \nabla \phi$  for some  $\phi \in W_0^{1,p}(\Omega)$  and  $\mathbf{z} \in V^p(\Omega)$ . Since  $\mathbf{u} \in V^p(\Omega) \subset W^p(\text{div}^0; \Omega)$ ,  $\mathbf{f} \in W^q(\text{div}^0; \Omega)$  and  $\nabla \times \nabla \phi = 0$  is well-defined for  $\phi \in W_0^{1,p}(\Omega)$ ,

$$\begin{aligned} (\partial_t \mathbf{u}, \mathbf{v})_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{v} \rangle_\Omega &= \left[ (\partial_t \mathbf{u}, \mathbf{z})_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{z} \rangle_\Omega \right] + (\partial_t \mathbf{u}, \nabla \phi)_\Omega + \langle \mathcal{P}(\mathbf{u}), \nabla \phi \rangle_\Omega \\ &= (\mathbf{f}, \mathbf{z})_\Omega + \underbrace{\frac{d}{dt} (\mathbf{u}, \nabla \phi)_\Omega}_{=0} + (\rho(\nabla \times \mathbf{u}) \nabla \times \mathbf{u}, \nabla \times \nabla \phi)_\Omega \\ &= (\mathbf{f}, \mathbf{z})_\Omega + \underbrace{(\mathbf{f}, \nabla \phi)_\Omega}_{=0} \\ &= (\mathbf{f}, \mathbf{v})_\Omega. \end{aligned}$$

We remark that the interchange of differentiation and integration was permitted by Theorem (2.27) of [15].  $\square$

**THEOREM 9.** *Consider a  $C^1$  simply connected bounded domain  $\Omega$  and a source term  $\mathbf{f} \in C(I; H(\text{div}^0; \Omega))$ . If  $\mathbf{u}$  and  $\mathbf{u}_h$  are the weak solutions to respectively (9) and (12), then there exists  $C > 0$  depending only on the shape regularity of  $\mathcal{T}_h$  such that for all  $\mathbf{v} \in W_0^p(\text{curl}; \Omega)$ ,*

$$(24) \quad \begin{aligned} &(\partial_t(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_\Omega + \langle \mathcal{P}(\mathbf{u}) - \mathcal{P}(\mathbf{u}_h), \mathbf{v} \rangle_\Omega \\ &\leq C((\eta_n + \eta_d) \|\mathbf{v}\|_{L^2(\Omega)} + (\eta_i + \eta_t) \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}), \end{aligned}$$

with

$$\begin{aligned} \eta_i^2 &:= \sum_{K \in \mathcal{T}_h} h_K^2 \|\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h)\|_{L^2(K)}^2, \\ \eta_d^2 &:= \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \cdot \partial_t \mathbf{u}_h\|_{L^2(K)}^2, \\ \eta_t^2 &:= \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \|\llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket\|_{L^2(F)}^2, \\ \eta_n^2 &:= \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \|\llbracket \gamma_n(\partial_t \mathbf{u}_h) \rrbracket\|_{L^2(F)}^2. \end{aligned}$$

*Proof.* Let  $\mathbf{u}$  satisfy (9) and  $\mathbf{u}_h$  satisfy (12). For any  $\mathbf{v} \in W_0^p(\text{curl}; \Omega)$  and  $\mathbf{v}_h \in V_{h,0} \subset W_0^p(\text{curl}; \Omega)$ ,

$$\begin{aligned} &(\partial_t(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_\Omega + \langle \mathcal{P}(\mathbf{u}) - \mathcal{P}(\mathbf{u}_h), \mathbf{v} \rangle_\Omega \\ &= \underbrace{\left[ (\partial_t \mathbf{u}, \mathbf{v} - \mathbf{v}_h)_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{v} - \mathbf{v}_h \rangle_\Omega \right]}_{=(\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_\Omega \text{ by Lemma 8}} - \left[ (\partial_t \mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_\Omega + \langle \mathcal{P}(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h \rangle_\Omega \right] \\ &\quad + \underbrace{\left[ (\partial_t \mathbf{u}, \mathbf{v}_h)_\Omega + \langle \mathcal{P}(\mathbf{u}), \mathbf{v}_h \rangle_\Omega \right]}_{=(\mathbf{f}, \mathbf{v}_h)_\Omega \text{ by Lemma 8}} - \underbrace{\left[ (\partial_t \mathbf{u}_h, \mathbf{v}_h)_\Omega + \langle \mathcal{P}(\mathbf{u}_h), \mathbf{v}_h \rangle_\Omega \right]}_{=(\mathbf{f}, \mathbf{v}_h)_\Omega \text{ since } \mathbf{u}_h \text{ satisfies (12)}} \\ &= (\mathbf{f} - \partial_t \mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_\Omega - \langle \mathcal{P}(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h \rangle_\Omega \\ &= \sum_{K \in \mathcal{T}} (\mathbf{f} - \partial_t \mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_K - (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h, \nabla \times (\mathbf{v} - \mathbf{v}_h))_K. \end{aligned}$$

When  $p \geq 2$  any  $\mathbf{v} \in W_0^p(\text{curl}; \Omega) \subset H_0(\text{curl}; \Omega)$  over bounded domains, and so the quasi-interpolant  $\mathbf{v}_h = \Pi \mathbf{v}$  of Theorem 6 is well-defined. Moreover, there exists  $\phi \in H_0^1(\Omega)$  and  $\mathbf{w} \in H_0^1(\Omega)^3$  for which  $\mathbf{v} - \Pi \mathbf{v} = \nabla \phi + \mathbf{w}$  and the estimates (18) and (19) hold. Thus, applying Green's formula (4) and (5) to our earlier rewriting of the left-hand side of (24), we find

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} (\mathbf{f} - \partial_t \mathbf{u}_h, \nabla \phi + \mathbf{w})_K - (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h, \nabla \times (\nabla \phi + \mathbf{w}))_K \\
&= \sum_{K \in \mathcal{T}_h} (\mathbf{f} - \partial_t \mathbf{u}_h, \mathbf{w})_K - (\nabla \cdot (\mathbf{f} - \partial_t \mathbf{u}_h), \phi)_K + (\gamma_n(\mathbf{f} - \partial_t \mathbf{u}_h), \gamma_0(\phi))_{\partial K} \\
&\quad - (\nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h), \mathbf{w})_K - (\gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h), \gamma_0(\mathbf{w}))_{\partial K} \\
&= \sum_{K \in \mathcal{T}_h} (\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h), \mathbf{w})_K - (\nabla \cdot (\mathbf{f} - \partial_t \mathbf{u}_h), \phi)_K \\
&\quad + \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \underbrace{\left( \llbracket \gamma_n(\mathbf{f}) \rrbracket, \gamma_0(\phi) \right)_E}_{=0, \text{ since at each } t \in I \mathbf{f} \in H(\text{div}; \Omega)} + \left( \llbracket \gamma_n(-\partial_t \mathbf{u}_h) \rrbracket, \gamma_0(\phi) \right)_E \\
&\quad + \left( \llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket, \gamma_0(\mathbf{w}) \right)_E \\
(25) \quad &= \sum_{K \in \mathcal{T}_h} R_i^K(\mathbf{u}_h; \mathbf{w}) + R_d^K(\mathbf{u}_h; \phi) + \sum_{F \in \mathcal{F}(\mathcal{T}_h)} R_n^F(\mathbf{u}_h; \phi) + R_t^F(\mathbf{u}_h; \mathbf{w}),
\end{aligned}$$

where the residuals are defined by

$$\begin{aligned}
R_i^K(\mathbf{u}_h; \mathbf{w}) &:= (\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h), \mathbf{w})_K, \\
R_d^K(\mathbf{u}_h; \phi) &:= -(\nabla \cdot (\mathbf{f} - \partial_t \mathbf{u}_h), \phi)_K, \\
R_n^F(\mathbf{u}_h; \gamma_0(\phi)) &:= (\llbracket \gamma_n(-\partial_t \mathbf{u}_h) \rrbracket, \gamma_0(\phi))_E \\
R_t^F(\mathbf{u}_h; \gamma_0(\mathbf{w})) &:= (\llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket, \gamma_0(\mathbf{w}))_E.
\end{aligned}$$

Indeed,  $R_i^K$  is the standard interior local residual term while  $R_n^F$  and  $R_t^F$  measure respectively the normal and tangential discontinuity of  $\gamma_n(-\partial_t \mathbf{u}_h)$  and  $\gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h)$  across neighbouring elements. We observe that at each  $t$ ,  $\mathbf{f} \in H(\text{div}^0; \Omega)$  implies that the first term in  $R_d^K$  satisfies  $(\nabla \cdot \mathbf{f}, \phi)_\Omega = 0$  but the second term  $\nabla \cdot \mathbf{u}_h$  vanishes only for first order Nédélec elements. The residual  $R_d^K$  therefore measures the defect in the divergence constraint at the discrete level, namely by

$$(26) \quad \sum_{K \in \mathcal{T}_h} R_d^K(\mathbf{u}_h; \phi) = \sum_{K \in \mathcal{T}_h} (\nabla \cdot \partial_t \mathbf{u}_h, \phi)_K.$$

Next, we proceed to estimate each term in the sum of (25) by using Holder's inequality, (18), and (19). We use the convention that the constant  $C$  may change from one line to the next and only depends on the shape-regularity of  $\mathcal{T}_h$ .

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} R_i^K(\mathbf{u}_h; \mathbf{w}) \leq \sum_{K \in \mathcal{T}_h} \|\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h)\|_{L^2(K)} \|\mathbf{w}\|_{L^2(K)} \\
& \leq C \sum_{K \in \mathcal{T}_h} h_K \|\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h)\|_{L^2(K)} \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)} \\
(27) \quad & \leq C \left( \sum_{K \in \mathcal{T}_h} h_K^2 \|\mathbf{f} - \partial_t \mathbf{u}_h - \nabla \times (\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h)\|_{L^2(K)}^2 \right)^{1/2} \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}
\end{aligned}$$

To bound the  $R_d^K$  term, we proceed in the same way

$$\begin{aligned}
R_d^K(\mathbf{u}_h; \mathbf{w}) &\leq \sum_{K \in \mathcal{T}_h} \|\nabla \cdot \partial_t \mathbf{u}_h\|_{L^2(K)} \|\phi\|_{L^2(K)} \\
&\leq C \sum_{K \in \mathcal{T}_h} h_K \|\nabla \cdot \partial_t \mathbf{u}_h\|_{L^2(K)} \|\mathbf{v}\|_{L^2(\omega_K)} \\
(28) \quad &\leq C \cdot \left( \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \cdot \partial_t \mathbf{u}_h\|_{L^2(K)}^2 \right)^{1/2} \cdot \|\mathbf{v}\|_{L^2(\Omega)}.
\end{aligned}$$

For the  $R_t^F(\mathbf{u}_h; \gamma_0(\mathbf{w}))$  term, we begin with a stability estimate. Using (11) and  $h_F \simeq h_K$  for shape-regular  $\mathcal{T}_h$ , we find

$$\begin{aligned}
\|\gamma_0(\mathbf{w})\|_{L^2(F)} &\leq C \left( h_F^{-1} \|\mathbf{w}\|_{L^2(K)}^2 + h_F \|\nabla \mathbf{w}\|_{L^2(K)}^2 \right)^{1/2} \\
&\leq C \left( h_F^{-1} h_K^2 \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)}^2 + h_F \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)}^2 \right)^{1/2} \\
&\leq C h_F^{1/2} \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)}.
\end{aligned}$$

Exploiting this last estimate, we proceed with

$$\begin{aligned}
\sum_{F \in \mathcal{F}(\mathcal{T}_h)} R_t^F(\mathbf{u}_h; \gamma_0(\mathbf{w})) &\leq \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \|\llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket\|_{L^2(F)} \|\gamma_0(\mathbf{w})\|_{L^2(F)} \\
&\leq C \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F^{1/2} \|\llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket\|_{L^2(F)} \|\nabla \times \mathbf{v}\|_{L^2(\omega_K)} \\
(29) \quad &\leq C \left( \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \|\llbracket \gamma_t(\rho(\nabla \times \mathbf{u}_h) \nabla \times \mathbf{u}_h) \rrbracket\|_{L^2(F)}^2 \right)^{1/2} \|\nabla \times \mathbf{v}\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly to the previous stability estimate, using (11) and the shape-regularity of  $\mathcal{T}_h$ , it is possible to show that

$$\|\gamma_0(\phi)\|_{L^2(F)} \leq C h_F^{1/2} \|\mathbf{v}\|_{L^2(\omega_K)}.$$

Applying this to the  $R_n^F(\mathbf{u}_h; \gamma_0(\phi))$  term, one finds

$$\begin{aligned}
\sum_{F \in \mathcal{F}(\mathcal{T}_h)} R_n^F(\mathbf{u}_h; \gamma_0(\phi)) &\leq \sum_{F \in \mathcal{F}(\mathcal{T}_h)} \|\llbracket \gamma_n(\partial_t \mathbf{u}_h) \rrbracket\|_{L^2(F)} \|\gamma_0(\phi)\|_{L^2(F)} \\
&\leq C \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F^{1/2} \|\llbracket \gamma_n(\partial_t \mathbf{u}_h) \rrbracket\|_{L^2(F)} \|\mathbf{v}\|_{L^2(\omega_K)} \\
(30) \quad &\leq C \left( \sum_{F \in \mathcal{F}(\mathcal{T}_h)} h_F \|\llbracket \gamma_n(\partial_t \mathbf{u}_h) \rrbracket\|_{L^2(F)}^2 \right)^{1/2} \|\mathbf{v}\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, combining (27)-(30), we have shown the desired result.  $\square$

Now we show the a posteriori error estimators in Theorem 9 are reliable in the following sense.

THEOREM 10. Let  $\mathbf{u}$ ,  $\mathbf{u}_h$  and  $\mathbf{f}$  as stated in Theorem 9 and denote the error as  $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$  and  $\mathbf{e}_0 = \mathbf{e}|_{t=0}$ , then for some positive constants  $C_1(p), C_2(p, T)$  so that

$$\begin{aligned} \sup_{s \in [0, T]} \|\mathbf{e}(s)\|_{L^2(\Omega)}^2 + C_1 \int_0^T \|\nabla \times \mathbf{e}(s)\|_{L^p(\Omega)}^p ds \\ \leq C_2 \left( \|\mathbf{e}_0\|_{L^2(\Omega)}^2 + \int_0^T \eta_n^2(s) + \eta_d^2(s) + \eta_i^q(s) + \eta_t^q(s) ds \right). \end{aligned}$$

*Proof.* Let  $\mathbf{v} = \mathbf{e} \in W_0^p(\text{curl}; \Omega)$  in (24). Recall the following inequality [11, eqn 24], for some  $C_p > 0$ ,

$$C_p |\mathbf{x} - \mathbf{y}|^p \leq (|\mathbf{x}|^{p-2} \mathbf{x} - |\mathbf{y}|^{p-2} \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Setting  $\mathbf{x} = \nabla \times \mathbf{u}$ ,  $\mathbf{y} = \nabla \times \mathbf{u}_h$  and integrating the inequality above, we obtain the coercivity estimate

$$(31) \quad C_p \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}^p \leq \langle \mathcal{P}(\mathbf{u}) - \mathcal{P}(\mathbf{u}_h), \mathbf{e} \rangle_\Omega.$$

Since both  $\mathbf{u}, \mathbf{u}_h \in L^p(\Omega) \subset L^2(\Omega)$ , and in fact  $\|\nabla \times \mathbf{e}\|_{L^2} \leq C \|\nabla \times \mathbf{e}\|_{L^p}$ , combining equation (24) with (31) and Young's inequality with  $\epsilon > 0$  gives,

$$(32) \quad \begin{aligned} \frac{d}{dt} \frac{1}{2} \|\mathbf{e}\|_{L^2(\Omega)}^2 + C_p \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}^p \\ \leq C \left( \frac{1}{2} (\eta_n^2 + \eta_d^2) + \frac{1}{2} \|\mathbf{e}\|_{L^2(\Omega)}^2 + \frac{1}{q\epsilon^q} (\eta_i^q + \eta_t^q) + \frac{\epsilon^p}{p} \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}^p \right). \end{aligned}$$

For sufficiently small  $\epsilon$ , inequality (32) implies that there exists positive constants  $C_1(p)$  and  $a(C, p)$  for which

$$\frac{d}{dt} \|\mathbf{e}\|_{L^2(\Omega)}^2 + C_1 \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}^p \leq a \left( \|\mathbf{e}\|_{L^2(\Omega)}^2 + \eta_n^2 + \eta_d^2 + \eta_i^q + \eta_t^q \right).$$

So by Gronwall's inequality,

$$\begin{aligned} \|\mathbf{e}(t)\|_{L^2(\Omega)}^2 + C_1 \int_0^t e^{a(t-s)} \|\nabla \times \mathbf{e}(s)\|_{L^p(\Omega)}^p ds \\ \leq e^{at} \|\mathbf{e}_0\|_{L^2(\Omega)}^2 + a \left( \int_0^t e^{a(t-s)} (\eta_n^2(s) + \eta_d^2(s) + \eta_i^q(s) + \eta_t^q(s)) ds \right) \\ \Rightarrow \|\mathbf{e}(t)\|_{L^2(\Omega)}^2 + C_1 \int_0^t \|\nabla \times \mathbf{e}(s)\|_{L^p(\Omega)}^p ds \\ (33) \quad \leq \max\{1, a\} e^{aT} \left( \|\mathbf{e}_0\|_{L^2(\Omega)}^2 + \int_0^T (\eta_n^2(s) + \eta_d^2(s) + \eta_i^q(s) + \eta_t^q(s)) ds \right), \end{aligned}$$

since  $1 \leq e^{a(t-s)} \leq e^{aT}$  for  $0 \leq s \leq t \leq T$ . Taking the supremum over all  $t \in [0, T]$  of equation (33) gives the desired result with  $C_2(p, T) = \max\{1, a\} e^{aT}$ .  $\square$

**5. A posteriori error estimate for AC loss.** For many engineering applications, the quantity of interest is the AC loss over one period  $T$ ,

$$Q(\mathbf{u}) := \frac{1}{T} \int_0^T \|\nabla \times \mathbf{u}(s)\|_{L^p(\Omega)}^p ds.$$

In particular, we wish to derive a posteriori error estimates for  $|Q(\mathbf{u}) - Q(\mathbf{u}_h)|$ . To do this, we first derive the following elementary estimate and subsequently use it to show the error for  $Q$  is related to the a posteriori error estimates derived previously.

LEMMA 11. *Assume  $1 \leq p$ , then for any positive functions  $x : [0, T] \rightarrow \mathbb{R}$ ,  $y : [0, T] \rightarrow \mathbb{R}$  uniformly bounded by  $M$  over  $[0, T]$ , we have*

$$\int_0^T |x(t)^p - y(t)^p| dt \leq pT^{1-\frac{1}{p}} M^{p-1} \left( \int_0^T |x(t) - y(t)|^p dt \right)^{1/p}.$$

*Proof.* For any  $t \in [0, T]$ , the mean value theorem implies there exists  $\xi(t) \in [0, M]$  satisfying

$$|x(t)^p - y(t)^p| = |x(t) - y(t)| \cdot p\xi(t)^{p-1} \leq pM^{p-1} |x(t) - y(t)|.$$

Thus, integrating over  $[0, T]$  gives,

$$\begin{aligned} \int_0^T |x(t)^p - y(t)^p| dt &\leq pM^{p-1} \int_0^T |x(t) - y(t)| dt \\ &\leq pT^{1-\frac{1}{p}} M^{p-1} \left( \int_0^T |x(t) - y(t)|^p dt \right)^{1/p}. \quad \square \end{aligned}$$

THEOREM 12. *Let  $\mathbf{u}$ ,  $\mathbf{u}_h$  be as stated in Theorem 9 and denote the error by  $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$  and  $\mathbf{e}_0 = \mathbf{e}|_{t=0}$ . Let  $M$  be stability bound for the weak formulation (9) and (12), then we have,*

$$\begin{aligned} |Q(\mathbf{u}) - Q(\mathbf{u}_h)| &\leq pT^{1-\frac{1}{p}} M^{p-1} \left( \int_0^T \|\nabla \times \mathbf{e}(s)\|_{L^p(\Omega)}^p ds \right)^{1/p} \\ &\leq \frac{C_2}{C_1} pT^{1-\frac{1}{p}} M^{p-1} \left( \|\mathbf{e}_0\|_{L^2(\Omega)}^2 + \int_0^T (\eta_n^2(s) + \eta_d^2(s) + \eta_i^q(s) + \eta_t^q(s)) ds \right)^{1/p} \end{aligned}$$

*Proof.* Let  $x(t) := \|\nabla \times \mathbf{u}\|_{L^p(\Omega)}$  and  $y(t) := \|\nabla \times \mathbf{u}_h\|_{L^p(\Omega)}$ .

Since  $0 \leq \|\nabla \times \mathbf{u}\|_{L^p(\Omega)} \leq M$ ,  $0 \leq \|\nabla \times \mathbf{u}_h\|_{L^p(\Omega)} \leq M$  for  $0 \leq t \leq T$ , Lemma (11) implies,

$$\begin{aligned} |Q(\mathbf{u}) - Q(\mathbf{u}_h)| &= \frac{1}{T} \left| \int_0^T (x(t)^p - y(t)^p) dt \right| \leq \frac{1}{T} \int_0^T |x(t)^p - y(t)^p| dt \\ (34) \quad &\leq pT^{1-\frac{1}{p}} M^{p-1} \left( \int_0^T |x(t) - y(t)|^p dt \right)^{1/p} \end{aligned}$$

Since  $|x(t) - y(t)| = \left| \|\nabla \times \mathbf{u}\|_{L^p(\Omega)} - \|\nabla \times \mathbf{u}_h\|_{L^p(\Omega)} \right| \leq \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}$  then by monotonicity of  $f(z) = z^p$ , we have  $|x(t) - y(t)|^p \leq \|\nabla \times \mathbf{e}\|_{L^p(\Omega)}^p$ . Thus, again by monotonicity of  $f(z) = z^{1/p}$ ,

$$(35) \quad \left( \int_0^T |x(t) - y(t)|^p dt \right)^{1/p} \leq \left( \int_0^T \|\nabla \times \mathbf{e}(s)\|_{L^p(\Omega)}^p ds \right)^{1/p}$$

Combining inequalities (34), (35) and Theorem 10 yield the desired result.  $\square$

**6. Numerical results.** We present numerical results in 2D supporting the reliability of the error estimators presented in Section 4. In the following, the  $p$ -curl problem is discretized in space using first order Nédélec elements and in time using the backward Euler method. While higher order time stepping schemes can be used, the discretization error is shown to be dominated by the spatial errors due to the low order approximation of first order Nédélec elements. The fully discrete formulation was implemented in Python using the FEniCS package [2]. For simplicity, we have scaled the units such that the material parameter  $\alpha$  is set to unity.

**6.1. Numerical verification of first order convergence.** We verify numerically first order convergence on the unit circle for a smooth radially symmetric solution  $\mathbf{u}(r, t) = r^a t^b \hat{\phi}$  with the forcing term  $\mathbf{f}(r, t) = (br^a t^{b-1} - ((a+1)t^b)^{p-1} r^{(a-1)(p-1)-1}) \hat{\phi}$ . Specifically, the constants  $a, b > 0$  are parameters to be chosen,  $r$  is the radial cylindrical coordinate and  $\hat{\phi}$  is the azimuthal unit vector. Note that by radial symmetry,  $\mathbf{u}(r, t)$  is necessarily divergence-free. For these tests, we have fixed  $p = 5$  and the final time  $T = 5e-3$ .

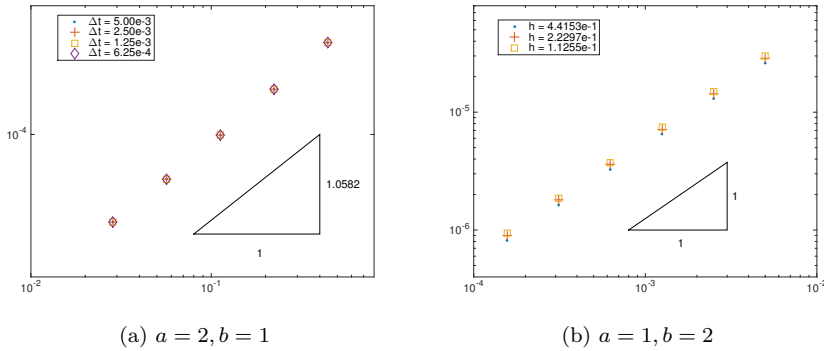
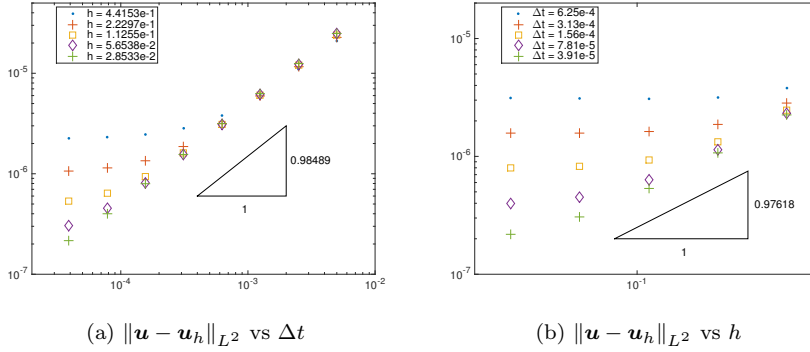


FIG. 1. Plot of  $\|\mathbf{u} - \mathbf{u}_h\|_{L^2}$  vs  $h$  and vs  $\Delta t$ , respectively.

For  $a = 1, b = 1$ , the solution is linear in both space and time. Since both first order Nédélec elements and backward Euler method are exact for linear functions, it was observed that the FE solution was accurate up to machine precision.

When  $a = 2, b = 1$ , the solution is quadratic in space and linear in time. Thus we expect to only have spatial error of first order in  $h$ , as shown in Figure 1a. Similarly, for the case  $a = 1, b = 2$ , we observed temporal error of first order in  $\Delta t$  in Figure 1b.

For  $a = 2, b = 2$ , the solution is quadratic in both space and time. From Figure 2a, first order error in  $\Delta t$  was observed in time when the mesh was sufficiently fine. Similarly from Figure 2b, first order error in  $h$  was observed in space when the time step size was sufficiently small.

FIG. 2. Plot of error versus  $\Delta t$  and  $h$  for  $a = 2, b = 2$ .

**6.2. Numerical verification of reliability of a posteriori error estimators.** Next, we numerically verify the reliability of the error estimators presented in Section 4. On the unit circle and  $t \in [0, 1]$ , we employed a radially symmetric moving front solution of the form  $\mathbf{u}(r, t) = h(r, t)\hat{\phi}$  with,

$$h(r, t) = \begin{cases} (r - 1 + t)^a, & r > 1 - t \\ 0, & r \leq 1 - t \end{cases},$$

where  $a \geq 1$  is a parameter to be chosen. It can be checked that the current density has the form  $\nabla \times \mathbf{u}(r, t) = j(r, t)\hat{\mathbf{z}}$  with

$$j(r, t) = \begin{cases} (r - 1 + t)^{a-1} \left( a + 1 - \frac{1-t}{r} \right), & r > 1 - t \\ 0, & r \leq 1 - t \end{cases}.$$

Thus, the corresponding forcing term is given by,

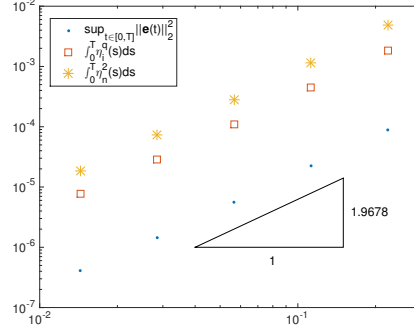
$$\mathbf{f}(r, t) = (h_t(r, t) - (p-1)j(r, t)^{p-2}j_r(r, t))\hat{\phi}.$$

The motivation for choosing this family of manufactured solutions originates from an exact analytical solution of Mayergoyz [26] of the  $p$ -curl problem in 1D. In particular, it is known that the parameter  $a = \frac{p-1}{p-2}$  for the 1D case and so  $a \approx 1$  for large values of  $p$ . Moreover, it can be seen that as  $a$  approaches 1, the current density  $j(r, t)$  has steeper gradients and converges pointwise to a discontinuous function. In fact for  $t < 1$ , it can be checked that  $j(r, t) \in W^{1,p}(\Omega)$  if and only if  $a > 2 - \frac{1}{p}$ .<sup>1</sup> Thus, for  $a$  close to 1, we do not expect the FE approximation using Nédélec elements to be accurate, since its interpolation error requires  $\nabla \times \mathbf{u}(r, t) = j(r, t)\hat{\mathbf{z}}$  to be at least a  $W^{1,p}(\Omega)$  function [13, Theorem 1.117]. For these reasons, we have focused on a case satisfying  $a > 2 - \frac{1}{p}$ . More specifically, we have fixed  $a = 3, p = 25, \Delta t = 5e-4$ .

The integration in time was computed numerically using the composite midpoint rule. Also note that, since the initial field  $\mathbf{u}_0(\mathbf{x}) = \mathbf{0} \in V_{h,0}^{(k)}$ , the initial error is identically zero. Moreover, recalling that first order Nédélec elements are element-wise divergence free, we omitted computing  $\eta_d$  as it is identically zero.

<sup>1</sup>Since  $j_r \sim s^{a-2}$  where  $s$  is the distance away from the front,  $j_r \in L^p(\Omega) \Leftrightarrow p(a-2) + 1 > 0$ .



FIG. 3. Comparison of error and estimators versus  $h$  at  $T = 4e-1$ .

In Figure 3, both the error in  $\sup_{s \in [0, T]} \|e(s)\|_{L^2(\Omega)}^2$  and estimators  $\int_0^T \eta_i^q(s) ds$  and  $\int_0^T \eta_n^2(s) ds$  from Theorem 10 are plotted for various mesh sizes  $h$ . Note that we have omitted showing  $\int_0^T \|\nabla \times e\|_{L^p(\Omega)}^p(s) ds$  and  $\int_0^T \eta_t^q(s) ds$  as their values were observed to be machine precision zero due to their small magnitude and their dependence on the exponent of  $p = 25$ . As illustrated, we observed quadratic order of convergence in  $h$  for both the error and estimators showing agreement of the reliability of the estimators. This is consistent with the first order convergence of Section 6.1, since the error quantity under consideration is squared with respect to the  $L^2$  norm.

In the absence of knowledge on the constants  $C_1$  and  $C_2$  from Theorem 10, we can still measure the reliability of the error estimators by the quantity  $\kappa^2$  defined as the ratio of estimators over the errors by,

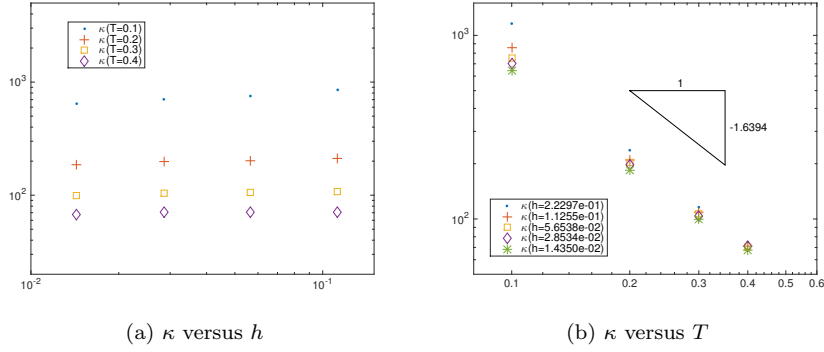
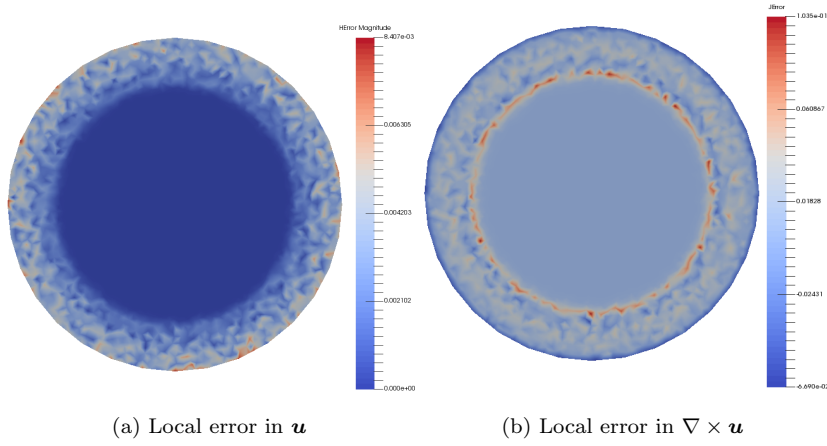
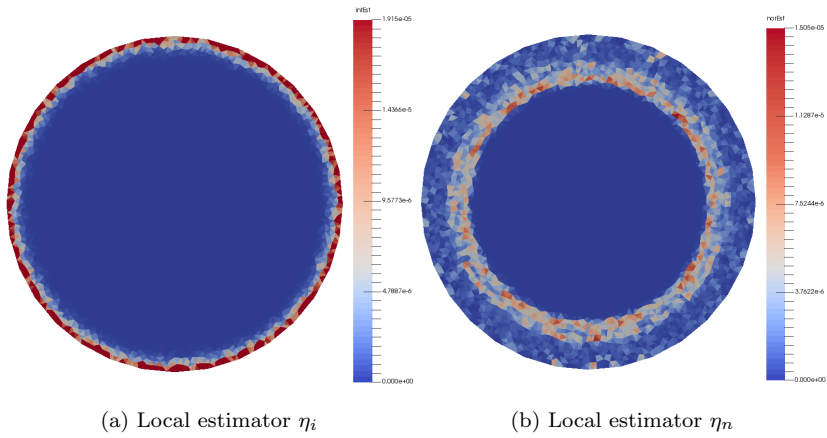
$$\kappa(\mathbf{u}, \mathbf{u}_h; T) = \frac{\int_0^T \eta_i^q(s) + \eta_t^q(s) + \eta_n^2(s) ds}{\sup_{s \in [0, T]} \|e(s)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla \times e(s)\|_{L^p(\Omega)}^p ds}.$$

Ideally, for efficient mesh adaptivity, one would like to have  $\kappa \approx 1$ . However, due to the unknown constants inherent in the present residual type error estimation and the dependence on  $T$  due to time integration, we can only expect  $\kappa$  to decrease with  $T$ . In particular, since the error estimators from Theorem 10 are reliable, then  $\kappa$  should be bounded below by the constant  $\frac{\min\{1, C_1(p)\}}{C_2(p, T)}$ , where  $C_2$  increases in the worst case exponentially with respect to  $T$ .

In Figure 4a,  $\kappa$  is shown to be largely independent of  $h$  and decreases with  $T$ . This suggests that the error estimators are comparable to the actual error up to a factor of  $\kappa$ . Moreover, from Figure 4b, we see that  $\kappa \sim T^{-1.64}$  which suggests that the exponential dependence on  $T$  for the constant  $C_2$  in Theorem 10 may be sharpened to  $\sim T^{1.64}$  in this case.

Finally, we look at a case for which  $\nabla \times \mathbf{u} \notin W^{1,p}(\Omega)$ . Specifically, we chose  $a = 1.6$  and  $p = 10$  so that  $a < 2 - \frac{1}{p}$ . The purpose here is to compare qualitatively between the error and estimators even in this nonsmooth case. As illustrated in Figure 5b and Figure 6b, the region where the local estimators  $\eta_n$  are largest agrees with regions where the sharp gradient occurs in the current density  $\nabla \times \mathbf{u}$ . Moreover, in Figure 5a and Figure 6a, the local estimators  $\eta_i$  identified the boundary region as where the increasing magnetic field  $\mathbf{u}$  was being applied.

<sup>2</sup>For stationary problems,  $\kappa$  is usually called the effectivity index of the error estimators.

FIG. 4. Comparison of  $\kappa$  versus  $h$  and  $T$ .FIG. 5. Local error of  $\mathbf{u}$  and  $\nabla \times \mathbf{u}$  at  $t = 0.272$ .FIG. 6. Local estimators  $\eta_i$  and  $\eta_n$  at  $t = 0.272$ .

**7. Conclusion.** This paper has presented an original a posteriori residual-based error estimator for a nonlinear wave-like propagation problem modeling strong variations in the magnetic field density inside high-temperature superconductors. The techniques used circumvent the non-conformity of the numerical approximations in a simple manner and the nonlinearities are handled using only coercive properties of the spatial operator, and without any linearization. Preliminary numerical results in two space dimensions indicate that the residuals are asymptotically exact, modulo up to a constant.

An important avenue for future research would be to develop error estimators which are both reliable and efficient. The work of Carstensen, Liu, and Yan on quasi-norms for the  $p$ -Laplacian appears to be the next natural step, given the similarities in the analytic framework underlying both problems [25, 8, 9] and the recent optimality results of Diening and Kreuzer on adaptive finite element methods for the  $p$ -Laplacian [12, 6]. Moreover, further investigation is needed concerning the efficiency for solving the nonlinear discrete problems arising from successive adaptive mesh based on such error estimators. At the moment, the optimal design of new high temperature superconducting devices is limited by the high computational cost of such simulations, and all means of improving this efficiency should be examined in hopes of removing this bottleneck.

**Appendix A. Non-homogeneous tangential boundary condition.** We can account for the non-homogeneous tangential boundary conditions on  $\partial\Omega$  by establishing a “Duhamel’s principle” for the  $p$ -curl problem. The novelty here is in the  $L^p$  treatment of the homogeneous auxiliary variables and in the nonlinearity.

Denote  $W^p(\text{curl}^0; \Omega) = \{\mathbf{v} \in W^p(\text{curl}; \Omega) : \nabla \times \mathbf{v} = 0\}$  as the  $L^p$  space of curl-free functions. It suffices to show the following:

**THEOREM 13.** *Let  $\Omega$  be a  $C^{1,1}$  bounded simply-connected domain in  $\mathbb{R}^3$  and let  $\mathbf{g} \in \gamma_t(W^p(\text{curl}; \Omega))$  with  $2 \leq p < \infty$ . For any  $\mathbf{u} \in W^p(\text{curl}; \Omega) \cap W^p(\text{div}^0; \Omega)$  with  $\gamma_t(\mathbf{u}) = \mathbf{g}$ , there exists a function  $\mathbf{u}_g \in W^p(\text{curl}^0; \Omega) \cap W^p(\text{div}^0; \Omega)$  with  $\gamma_t(\mathbf{u}_g) = \mathbf{g}$  and a function  $\hat{\mathbf{u}} \in V^p(\Omega)$  such that  $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_g$ .*

Indeed, if such decomposition exists, since  $\mathbf{u}_g$  is curl- and divergence-free, the non-homogeneous  $p$ -curl problem reduces to the homogeneous  $p$ -curl problem,

$$(36) \quad \begin{aligned} \partial_t \hat{\mathbf{u}} + \nabla \times [\rho(\nabla \times \hat{\mathbf{u}}) \nabla \times \hat{\mathbf{u}}] &= \mathbf{f} - \partial_t \mathbf{u}_g, & \text{in } I \times \Omega, \\ \nabla \cdot \hat{\mathbf{u}} &= 0, & \text{in } I \times \Omega, \\ \hat{\mathbf{u}}(0, \cdot) &= \mathbf{u}_0(\cdot) - \mathbf{u}_g(0, \cdot), & \text{in } \Omega, \\ \mathbf{n} \times \hat{\mathbf{u}} &= 0, & \text{on } I \times \partial\Omega. \end{aligned}$$

*Proof.* Given a function  $\mathbf{g} \in \gamma_t(W^p(\text{curl}; \Omega))$ , we construct  $\mathbf{u}_g \in W^p(\text{curl}^0; \Omega) \cap W^p(\text{div}^0; \Omega)$  in three main steps.

First, let  $\tilde{\mathbf{u}}_g \in W^p(\text{curl}; \Omega)$  be such that  $\gamma_t(\tilde{\mathbf{u}}_g) = \mathbf{g}$ . Such  $\tilde{\mathbf{u}}_g$  exists by the surjectivity of the image space  $\gamma_t(W^p(\text{curl}; \Omega))$ .

Second, let  $v \in W_0^{1,p}(\Omega)$  be the solution to the problem:

$$(37) \quad (\nabla v, \nabla \psi)_\Omega = (\tilde{\mathbf{u}}_g, \nabla \psi)_\Omega, \quad \forall \psi \in W_0^{1,q}(\Omega).$$

Such a function  $v$  exists if the following two conditions hold [13]:

$$(38) \quad 0 < \inf_{0 \neq \phi \in W_0^{1,p}(\Omega)} \sup_{0 \neq \psi \in W_0^{1,q}(\Omega)} \frac{(\nabla \phi, \nabla \psi)_\Omega}{\|\phi\|_{W_0^{1,p}(\Omega)} \|\psi\|_{W_0^{1,q}(\Omega)}},$$

and if for all  $\phi \in W_0^{1,p}(\Omega)$ ,

$$(39) \quad (\nabla \phi, \nabla \psi)_\Omega = 0, \quad \forall \psi \in W_0^{1,q}(\Omega) \quad \Rightarrow \quad \phi = 0.$$

We first show the inf-sup condition. From the Helmholtz decomposition of Theorem 3, for  $\mathbf{v} \in W_0^{1,q}(\Omega)^3$ , there exists  $\mathbf{z}_v \in V^q(\Omega)$  and  $\phi_v \in W_0^{1,q}(\Omega)$  such that  $\mathbf{v} = \mathbf{z}_v + \nabla \phi_v$  with  $\|\mathbf{z}_v\|_{L^q} + \|\nabla \phi_v\|_{L^q} \leq C \|\mathbf{v}\|_{L^q}$  for some constant  $C > 0$ . In particular, for any  $\phi \in W_0^{1,p}(\Omega)$ ,  $(\nabla \phi, \mathbf{z}_v)_\Omega = 0$ . This implies that for any  $\phi \in W_0^{1,p}(\Omega)$ ,

$$\begin{aligned} \|\phi\|_{W_0^{1,p}(\Omega)} &= \sup_{0 \neq \mathbf{v} \in L^q(\Omega)} \frac{(\nabla \phi, \mathbf{v})_\Omega}{\|\mathbf{v}\|_{L^q(\Omega)}} = \sup_{0 \neq \mathbf{v} \in L^q(\Omega)} \frac{(\nabla \phi, \nabla \phi_v)_\Omega}{\|\mathbf{v}\|_{L^q(\Omega)}} \\ &\leq C \sup_{0 \neq \mathbf{v} \in L^q(\Omega)} \frac{(\nabla \phi, \nabla \phi_v)_\Omega}{\|\nabla \phi_v\|_{L^q(\Omega)}} \leq C \sup_{0 \neq \psi \in W_0^{1,q}(\Omega)} \frac{(\nabla \phi, \nabla \psi)_\Omega}{\|\nabla \psi\|_{L^q(\Omega)}}. \end{aligned}$$

Since the norm  $\|\nabla \psi\|_{L^q(\Omega)}$  is equivalent to  $\|\psi\|_{W_0^{1,q}(\Omega)}$  for  $\psi \in W_0^{1,q}(\Omega)$ , dividing the above inequality by  $\|\phi\|_{W_0^{1,p}(\Omega)}$  and taking the infimum over  $\phi \in W_0^{1,p}(\Omega)$  shows the inf-sup condition (38) is satisfied.

We now explain why condition (39) also holds. For  $\psi = \phi \in W_0^{1,2}(\Omega) \subset W_0^{1,q}(\Omega)$ , by Poincaré's inequality the condition  $0 = (\nabla \phi, \nabla \psi)_\Omega = \|\nabla \phi\|_{L^2(\Omega)}^2$  implies that  $\phi = 0$  almost everywhere. Thus, a unique solution  $v \in W_0^{1,p}(\Omega)$  to (37) exists.

Third, let  $\mathbf{w} \in V^p(\Omega)$  be the solution to the problem:

$$(40) \quad (\nabla \times \mathbf{w}, \nabla \times \psi)_\Omega = (-\nabla \times \tilde{\mathbf{u}}_g, \nabla \times \psi)_\Omega, \quad \forall \psi \in V^q(\Omega).$$

Similarly, such a function  $\mathbf{w}$  exists if the following two conditions hold:

$$(41) \quad 0 < \inf_{0 \neq \phi \in V^p(\Omega)} \sup_{0 \neq \psi \in V^q(\Omega)} \frac{(\nabla \times \phi, \nabla \times \psi)_\Omega}{\|\phi\|_{V^p(\Omega)} \|\psi\|_{V^q(\Omega)}},$$

and if for all  $\phi \in V^p(\Omega)$ ,

$$(42) \quad (\nabla \times \phi, \nabla \times \psi)_\Omega = 0, \quad \forall \psi \in V^q(\Omega) \quad \Rightarrow \quad \phi = 0.$$

By Lemma 5.1 of [3], the inf-sup condition (41) is satisfied. Moreover, since for  $\psi = \phi \in V^2(\Omega) \subset V^q(\Omega)$ ,  $0 = (\nabla \times \phi, \nabla \times \psi)_\Omega = \|\nabla \times \phi\|_{L^2(\Omega)}^2$  implies  $\phi = 0$  a.e. by the equivalence of the semi-norm on  $V^p(\Omega)$ ; see Corollary 3.2 of [3]. Hence, a unique solution  $\mathbf{w} \in V^p(\Omega)$  to (40) exists.

Combining these three functions, we define

$$\mathbf{u}_g := \mathbf{w} + \tilde{\mathbf{u}}_g - \nabla v \in W^p(\text{curl}; \Omega).$$

Note that  $\gamma_t(\mathbf{u}_g) = \gamma_t(\mathbf{w}) + \gamma_t(\tilde{\mathbf{u}}_g) - \gamma_t(\nabla v) = \mathbf{g}$ , since  $\mathbf{w} \in V^p(\Omega)$  and  $v \in W_0^{1,p}(\Omega)$ . Since  $\mathbf{w} \in V^p(\Omega)$  is divergence-free,  $\nabla \cdot \mathbf{u}_g = \nabla \cdot (\tilde{\mathbf{u}}_g - \nabla v) = 0$  as  $v$  satisfies (37). Moreover,  $\nabla \times \mathbf{u}_g = \nabla \times (\mathbf{w} + \tilde{\mathbf{u}}_g) = 0$  since  $\mathbf{w}$  satisfies (40); i.e.  $\mathbf{u}_g \in W^p(\text{curl}^0; \Omega) \cap W^p(\text{div}^0; \Omega)$ . Thus,  $\mathbf{u}_g \in W^p(\text{curl}^0; \Omega) \cap W^p(\text{div}^0; \Omega)$  with  $\gamma_t(\mathbf{u}_g) = \mathbf{g}$ .

Finally, defining  $\hat{\mathbf{u}} := \mathbf{u} - \mathbf{u}_g \in W^p(\text{curl}; \Omega) \cap W^p(\text{div}^0; \Omega)$  and noting  $\gamma_t(\hat{\mathbf{u}}) = \gamma_t(\mathbf{u}) - \gamma_t(\mathbf{u}_g) = 0$ ,  $\mathbf{u} - \mathbf{u}_g \in V^p(\Omega)$ . This shows that  $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_g$  as claimed.  $\square$

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